Trees in tournaments

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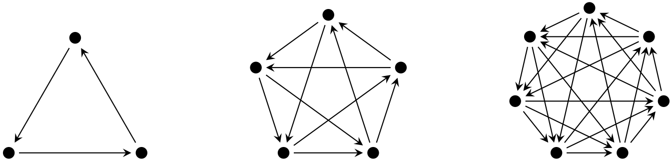
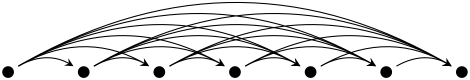
Tournament

tournament = Orientation of a complete graph.



transitive tournament = tournament with no directed cycle
TT_k = transitive tournament of order k.



Unavoidability

n-unavoidable = contained in every tournament of order n

unvd(D) : unavoidability = minimum n s.t. D is n-unavoidable.

unvd(D) $< +\infty$ if and only if D is acyclic.

- ▶ unavoidable $\Rightarrow$ contained in TT_n $\Rightarrow$ no directed cycle
- ◀ every acyclic digraph of order k is contained in TT_k.

Upper bounds on unvd(TT_k)

$$\text{unvd}(TT_k) \leq 2\,\text{unvd}(TT_{k-1})$$

[[*Proof* : A tournament of order $2\,\text{unvd}(TT_{k-1})$ contains a vertex with $d^+ \geq \text{unvd}(TT_{k-1})$.]]

Corollary $\text{unvd}(TT_k) \leq 2^{k-1}$.

$\text{unvd}(TT_1) = 1$, $\text{unvd}(TT_2) = 2$, $\text{unvd}(TT_3) = 4$, and $\text{unvd}(TT_4) = 8$ (because of Paley tournament).
Reid and Parker, 1970 : $\text{unvd}(TT_5) = 14$, $\text{unvd}(TT_6) = 28$.
Sanchez-Flores, 1994 : $\text{unvd}(TT_7) = 54$.

Corollary $\text{unvd}(TT_k) \leq 54 \times 2^{k-7}$ (for $k \geq 7$).

Lower bounds on unvd(TT_k)

Theorem (Erdős and Moser, 1964) unvd$(TT_k) > 2^{(k-1)/2}$.

[[*Proof* : Random tournament T on $n = 2^{(k-1)/2}$ vertices.
Probability that $T\langle v_1, \dots, v_k\rangle$ is transitive with hamiltonian dipath $(v_1, \dots, v_k)$ is $\left(\frac{1}{2}\right)^{\binom{k}{2}}$.
Expected number of transitive tournaments : $\frac{n!}{(n-k)!}\left(\frac{1}{2}\right)^{\binom{k}{2}}$

$$< n^k \left(\frac{1}{2}\right)^{\binom{k}{2}} \leq 1.$$

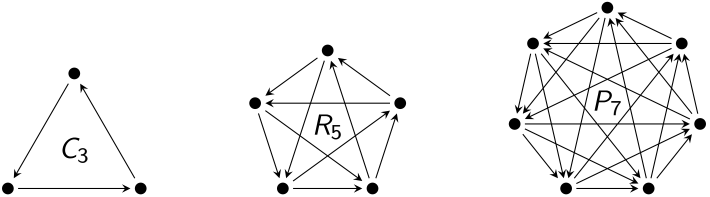
Simple Moment Method, n-tournament with no TT_k.]]

Theorem For every $C > 1$, $C \times \text{unvd}(TT_k) > 2^{(k+1)/2}$ if n is large enough.

[[Use Local Lemma]]

Oriented paths in tournament

$\vec{P}_n$: directed path on n vertices.
Theorem (Redei, 1934) Every tournament has a directed Hamiltonian path. $\text{unvd}(\vec{P}_n) = n$.



Theorem (H. and Thomassé, 2000). $\text{unvd}(P) = |P|$ if $|P| \geq 8$.
T tournament, P oriented path with $|T| = |P|$.
T contains P unless $T \in \{C_3, R_5, P_7\}$ and P is antidirected.

Oriented cycles in tournament

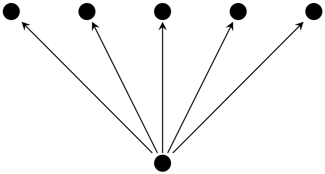
Theorem (Thomason, 1986).
If C is a non-directed cycle with $|C| \geq \mathbf{2^{128}}$, then $\mathrm{unvd}(C) = |C|$.

Theorem (H. , 2000).
If C is an non-directed cycle with $|C| \geq \mathbf{68}$, then $\mathrm{unvd}(C) = |C|$.

Conjecture
If C is an non-directed cycle with $|C| \geq \mathbf{9}$, then $\mathrm{unvd}(C) = |C|$.

Oriented trees in tournament

Conjecture (Sumner, 1972).
If T is an oriented tree or order n, then $\text{unvd}(T) \leq 2n - 2$.



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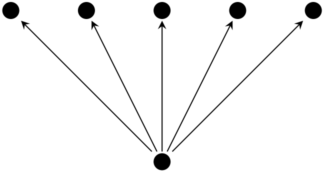

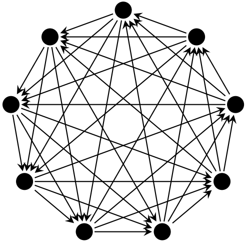

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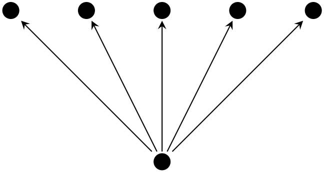

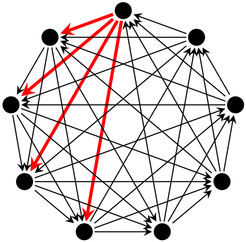

Universal digraphs

Theorem (Gallai 1968, Hasse 1964, Roy 1967, Vitaver 1962)
If $\chi(D) \geq n$, then D contains a directed path of order n.

n-universal = contained in every digraph D with $\chi(D) \geq n$.

Theorem (Erdős, 1959)
For all k, g, there are graphs with $\chi \geq k$ and girth $\geq g$.

universal digraph must be the orientation of a forest.

Theorem (Burr, 1980)
Every oriented forest of order n is n^2-universal.

Addario-Berry et al.2013 improved to $\frac{1}{2}n^2 - \frac{1}{2}n + 1$-universal.

Conjecture (Burr, 1982)
Every oriented forest of order n is $(2n-2)$-universal.

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(Häggkvist and Thomason, 1991)	$12n$	$(4 + o(1))n$
(H. and Thomassé, 2000)	$\frac{7}{2}n - \frac{5}{2}$	
(El Sahili, 2004)	$3n - 3$	
(Kühn, Mycroft and Osthus, 2011)	$2n - 2$ for n large	.

Theorem (H. and Thomassé, 2000).
If A is an arborescence, then $\text{unvd}(A) \leq 2|A| - 2$.

Beyond Sumner's conjecture

Conjecture (H. and Thomassé, 2000).
If T is an oriented tree of order n with k leaves, then

$$\text{unvd}(T) \leq n + k - 1.$$

Evidences : True for $k \leq 3$. (Ceroi and H., 2004).

True for a large class of trees. (H. 2002) .

$\text{unvd}(T) \leq n + 2^{512k^3}$. (Häggkvist and Thomason, 1991)

Our results

Theorem (Dross and H. , 2018).
If A is an out-arborescence of order n with k out-leaves,
then $\text{unvd}(A) \leq n + k - 1.$

Theorem (Dross and H. , 2018).
If T is a tree of order n with k leaves, then

$$\text{unvd}(T) \leq \begin{cases} \frac{3}{2}n + \frac{3}{2}k - 2 \end{cases}$$

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Median orders

median order : $(v_1, v_2, \ldots, v_n)$ s.t. $|\{(v_i, v_j) : i < j\}|$ is maximum.

Proposition : If $(v_1, v_2, \ldots, v_n)$ is a median order of T, then

(M1) $(v_i, v_{i+1}, \ldots, v_j)$ is a median order of $T\langle\{v_i, v_{i+1}, \ldots, v_j\}\rangle$;

(M2) v_i dominates at least half of the vertices $v_{i+1}, \ldots, v_j$, and
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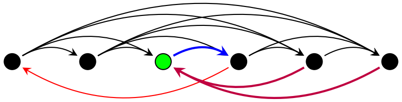

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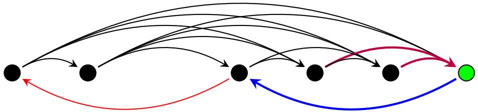

Arborescences : the greedy procedure

A out-arborescence with root r, n nodes, k out-leaves.
$(v_1, \ldots, v_m)$ median order of T with $|T| = m = n + k - 1$.

Set $\phi(r) = v_1$.

For $i = 1$ to m, do

- if v_i is not hit, skip; v_i is failed $(v_i \in F)$
- if v_i is hit, let $a_i = \phi^{-1}(v_i)$;
  assign the $|N^+(a_i)|$ first not yet hit out-neighbours of v_i in $\{v_{i+1}, \ldots, v_m\}$ to the sons of a_i (according to some predefined order);

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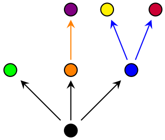

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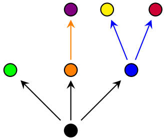

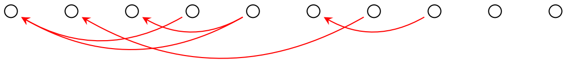

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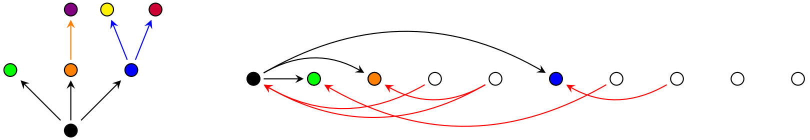

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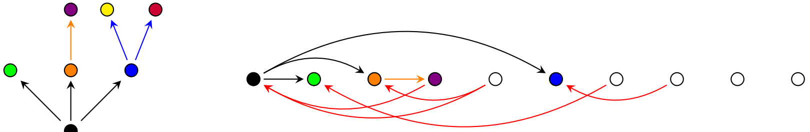

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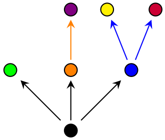
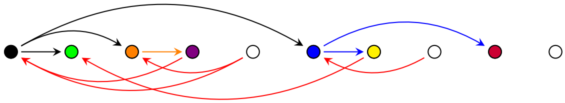




Arborescences : analysis

node a is active for i if $\phi(a) \in \{v_1, \ldots, v_i\}$ and it has a son b that is not embedded in $\{v_1, \ldots, v_i\}$.
For $v_i \in F$, let ℓ_i be the largest index such that a_{ℓ_i} is active for i.
Set $I_i = \{v_{\ell_{i+1}}, \ldots, v_i\}$.

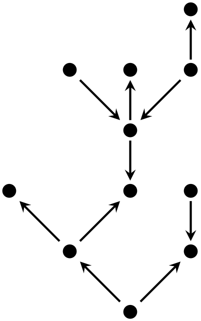
Claim 1: If $v_i \in F$, then $|I_i \cap F| \leq |I_i \cap \phi(L)|$. $\quad L = \{\text{out-leaves}\}$.

Claim 2: If $v_i, v_j \in F$, then either $I_i \cap I_j = \emptyset$, or $I_i \subseteq I_j$, or $I_j \subseteq I_i$.

M: the set of indices i such that $v_i \in F$ and I_i is maximal for inclusion.
$|F| = \sum_{i \in M} |I_i \cap F| \leq \sum_{i \in M} |I_i \cap \phi(L)| \leq |\phi(L)| = |L| \leq k - 1.$

unvd(A) $\leq \frac{3}{2}n + \frac{3}{2}k - 2$: the downward forest

A : tree rooted in r with n nodes and k leaves.



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upward arcs : arcs directed away from the root
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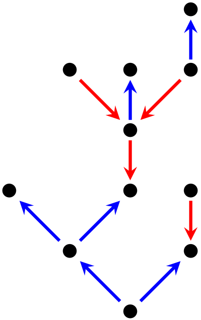

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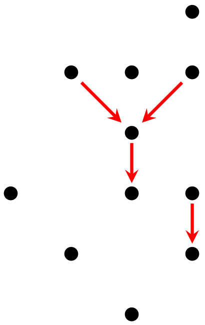
downward forest : subdigraph induced by the downward arcs



unvd$(A) \leq \frac{3}{2}n + \frac{3}{2}k - 2$: the lemma

$\mathcal{C}_r^{\downarrow}$: set of components of the downward forest

$$\gamma_r^{\downarrow} = \sum_{C \in \mathcal{C}_r^{\downarrow}} (|V(C)| + |L^-(C)| - 2)$$

Lemma If r is a source, then A is $(n + k - 1 + \gamma_r^{\downarrow})$-unavoidable.

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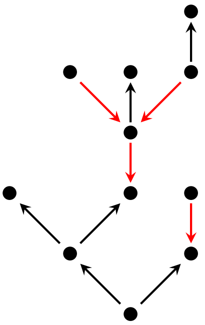

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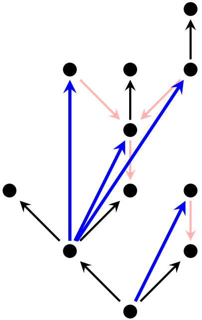

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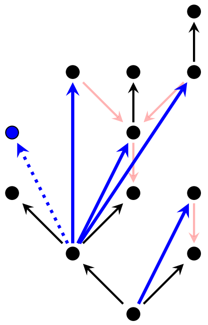

n_i vertices; k_i in-leaves $\quad k_i - 1$ new vertices

$\mathrm{unvd}(A) \leq \frac{3}{2}n + \frac{3}{2}k - 2$: concluding

A : tree rooted in r with n nodes and k leaves.

$\gamma_r^{\downarrow} = \sum_{C \in \mathcal{C}_r^{\downarrow}} (|V(C)| + |L^-(C)| - 2)$

Pick r such that $\min(\gamma_r^{\uparrow}, \gamma_r^{\downarrow})$ is minimum.
W. l. o. g. this minimum is attained by $\gamma_r^{\downarrow}$.

$\gamma_r^{\uparrow} + \gamma_r^{\downarrow} \leq n + k - 2$, so $\gamma_r^{\downarrow} \leq \frac{1}{2}(n+k) - 1$

r is source.

So, by the Lemma, A is $(\frac{3}{2}n + \frac{3}{2}k - 2)$-unavoidable.

unvd$(A) \leq n + O(k^2)$: cutting the tree

Theorem (Thomason, 1986)
P non-directed path of order n with first and last block of length 1.
T tournament of order $n+2$ and $X, Y \subseteq V(T)$, $|X|, |Y| \geq 2$.
If $P \neq \pm(1,1,1)$, then there is a copy of P in T with origin in X and terminus in Y.

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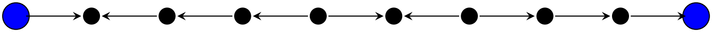

unvd$(A) \leq n + O(k^2)$: cutting the tree

Theorem (Thomason, 1986)
P non-directed path of order n with first and last block of length 1.
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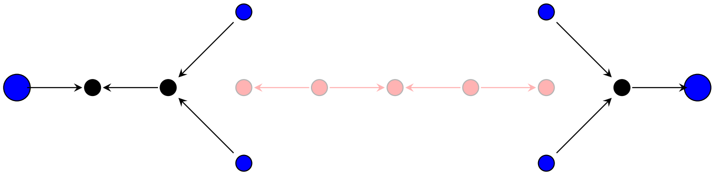


unvd(A) $\leq n + O(k^2)$: reduction to stubs

stub : tree such that

(i) every inner segment has at most three blocks; moreover, if it has three blocks then its first and third block have length 1, and if it has two blocks then one of them has length 1.

(ii) every outer segment has length 1.

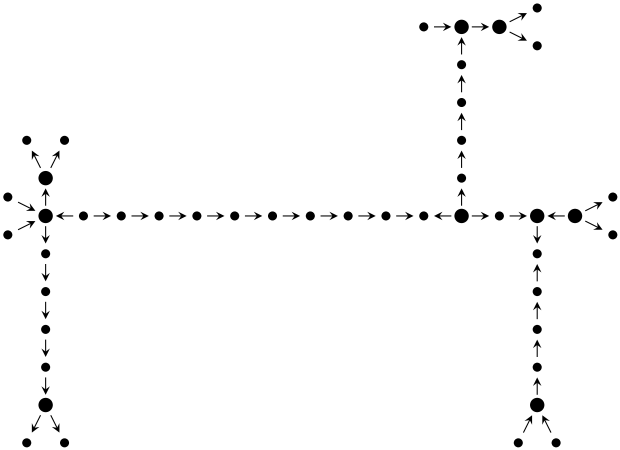
Lemma

Every stub of order n with k leaves is $(n + f(k))$-unavoidable,

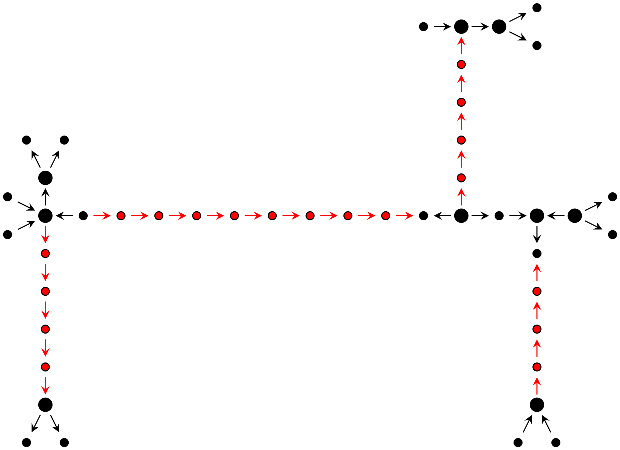
$$\Downarrow$$

every tree of order n with k leaves is $(n + \max\{f(2k - 2b) + b \mid 0 \leq b \leq k - 3\})$-unavoidable.

unvd(A) $\leq n + O(k^2)$: organizing the stubs



unvd$(A) \leq n + O(k^2)$: organizing the stubs



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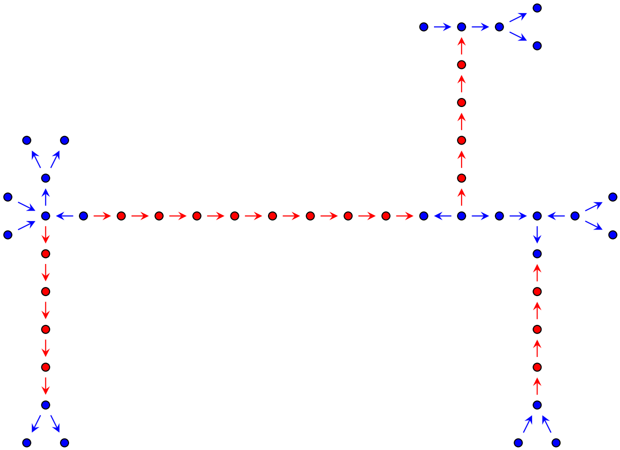
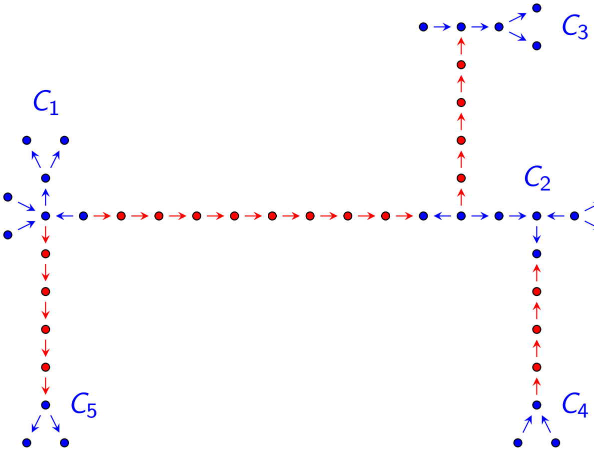


$\mathrm{unvd}(A) \leq n + O(k^2)$: organizing the stubs



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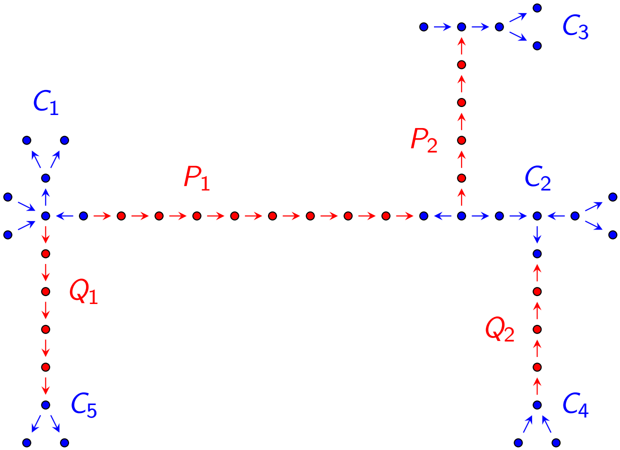
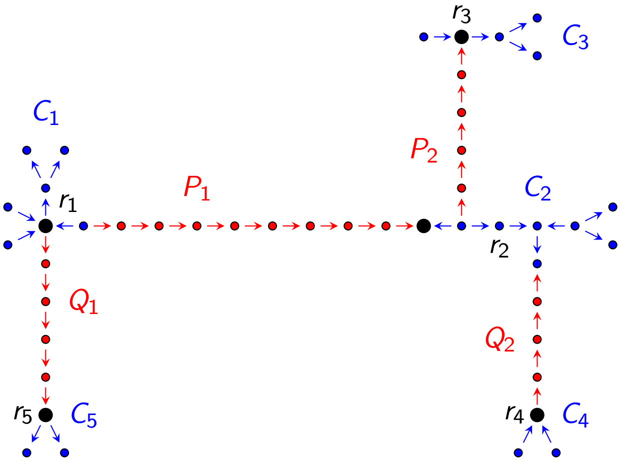


$\text{unvd}(A) \leq n + O(k^2)$: organizing the stubs



Stubs: the rabbit hop

Lemma: $m \geq 4k$, $(v_1, \ldots, v_m)$ median order of T.
There are k internally disjoint 2-dipaths from v_1 to $\{v_{m-4k+2}, \ldots, v_m\}$.